Principles of integrability by examples/applications

18-9-2019, Pisa PRIN Kick-off Meeting

Davide Fioravanti (INFN-Bologna)

series of paper with M.Rossi,JE Bourgine, D.Gregori, A. Bonini, H.Poghossian,….

- Sketch of a PLAN in integrable words :
- 1) Motivations: different research topics (e.g. WL string minimal area) lead us to Thermodynamic Bethe Ansatz in the ODE/IM perspective
- 2) Traditional (scattering) way to TBA (I way)
- 3) ODE/IM and PDE/IM: functional and integral eqs. (II way to TBA)
- 4) OPE or Form Factor Series for null polygonal WLs re-sums to TBA: III way

Some motivations and perspectives

- General wall-crossing (jumping) formulae (Donaldson-Thomas invariants) e.g. by Kontsevich-Soibelman have taken a very effective form for BPS states (compactified theories) thanks to Gaiotto-Moore-Neitzke (2008)
- $$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(\zeta) \exp\left[-\frac{1}{4\pi i}\sum_{\gamma'}\Omega(\gamma';u)\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\sigma(\gamma')\mathcal{X}_{\gamma'}(\zeta'))\right]. \quad (5.13)$$
- which are nothing but TBA EQS. In fact more that one year later, enriched perspective
- *Note added Nov. 20, 2009*:

  It was pointed out to us some time ago by A. Zamolodchikov that one of the central results of this paper, equation (5.13), is in fact a version of the Thermodynamic Bethe Ansatz [45]. In this appendix we explain that remark. Another relation between four-dimensional super Yang-Mills theory and the TBA has recently been discussed by Nekrasov and Shatashvili [46].

  The TBA equations for an integrable system of particles a with masses m_a, at inverse temperature β, with integrable scattering matrix $S_{ab}(\theta-\theta')$, where θ is the rapidity, are

  $$\epsilon_a(\theta) = m_a\beta\cosh\theta - \sum_b\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\phi_{ab}(\theta-\theta')\log(1+e^{\beta\mu_b-\epsilon_b(\theta')}) \quad \text{(E.1)}$$

  where $\phi_{ab}(\theta) = -i\frac{\partial}{\partial\theta}\log S_{ab}(\theta)$. Here the scattering matrix is diagonal, that is, the soliton creation operators obey $\Phi_a(\theta)\Phi_b(\theta') = S_{ab}(\theta-\theta')\Phi_b(\theta')\Phi_a(\theta)$.

circumference R

more general

- Hitchin systems: the same mathematical problem as for minimal string area for gluon scattering amplitudes/Wilson loops (null, polygonal) in N=4 SYM
- Benefit for exchange of ideas between these fields and from integrability ideas (non-perturbative, exact, ect) which makes clear the following:
- The general phenomenon on the background is the so-called linear Ordinary Differential Equation/Integrable Model (ODE/IM) correspondence (CFTs), possibly extended to linear PDE (Massive QFTs)
- Recently we proposed an advance (different ODE) which identifies NS (SW with one Omega background) periods with integrable quantities T,Q: functional and integral eqs. Pandora box? I will give you a flavour.
- We re-summed the OPE (FF) series of Wl (collinear limit) to TBA: why?
- Before that, let us recall the original physics of TBA.

The Thermodynamic Bethe Ansatz

- Evolution of Zamolodchikov's idea to non-relativistic theories, where the scattering matrix does not change (as depends on difference of rapidities which are all shifted).
- A cylinder (p.b.c.: torus) of very large height R (time) and circumference L (space) may be seen in the other way around:

$$L(space) \leftrightarrow R(time) \quad p \leftrightarrow E \qquad ABA(direct) \rightarrow \tilde{ABA}(mirror)$$

i.e. analytic continuation which entails the same partition function

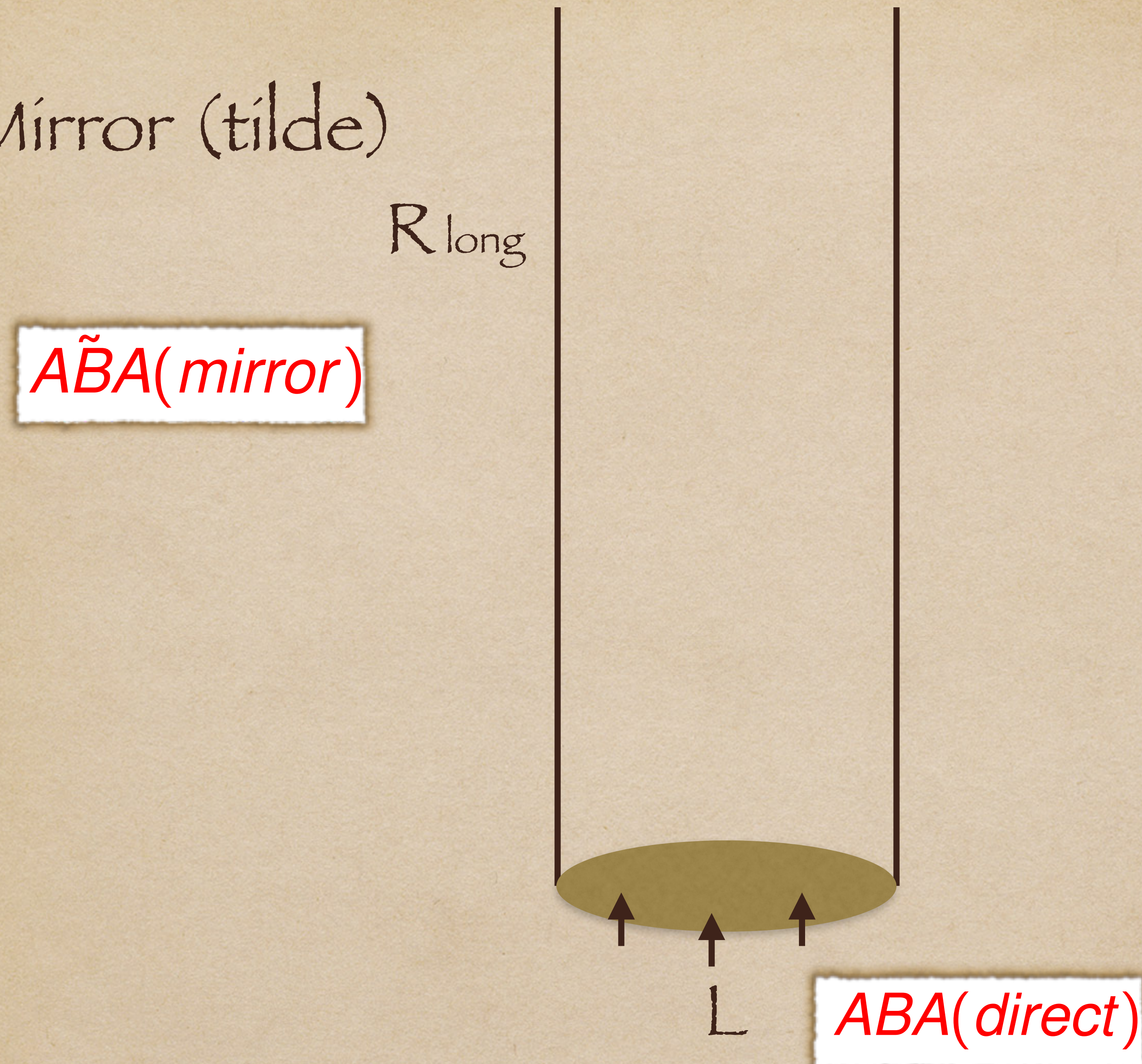
$$Z_{direct}(L,R) = \tilde{Z}_{mirror}(L,R).$$

- Advantage: asymptotic BA exact in the mirror theory at $R=\infty$, then thermodynamics for minimal free energy at 'temperature' $T = 1/L$

$$\exp[-RE_0(L)] = \exp[-RL\tilde{f}_{min}(L)], \;\; R \to \infty$$

furnishes the ground state energy of direct (string/gauge) theory $E_0(L)$.
- Infinite system of non-linear (real) integral equations and $E(L)$ is a non-linear functional on the real rapidity u summed up on infinite pseudoenergies $\epsilon_Q(u)$ (massive nodes).

Mirror (tilde)



Direct: Tr(Z^L) or Tr(....)+....

Vacuum/Excited states Thermodynamic Bethe Ansatz

- Vacuum equations of the form

$$\epsilon_a(u) = \mu_a + \tilde{e}_a(u) - \sum_b \int dv \;\; K_{a,b}(u,v) \ln(1 + e^{-\epsilon_b(v)})$$

  with mirror energy $\tilde{e}_a(u)$ as driving term and scattering factors

$$K_{a,b}(u,v) \propto \partial_v \ln S_{a,b}(u,v)$$

- Excited states $E(L)$ are connected to the vacuum by analytic continuation in some parameter (*e.g.* μ_a and L) $\Rightarrow$ additional inhomogeneous terms in the equations $\sum_i \ln S_{a,b}(u, u_i)$ depending on TBA complex singularities u_i:

$$e^{-\epsilon_a(u_i)} = -1$$

  these are the exact Bethe roots (with wrapping).
- $\Rightarrow$ Delicate and massive numerical work for analytic continuation.

Excited states via the Y-system

- Alternative route: for simpler integrable theories (like quantum Sine-Gordon) we proposed and checked all the states - including the ground state! - must satisfy the same functional equations, the so-called Y-system:
$$Y_a(u) \equiv e^{-\epsilon_a(u)}.$$
In a nutshell, we loose the information concerning the inhomogeneous terms as they are zero-modes of the 'TBA-operator' (a multi-shift operator with incidence matrix), *i.e.* $\ln S_{a,b}(u, u_i)$ (sort of solution of Y-system). Universal, but we recover the specific forcing term/state by behaviour at $u = \pm\infty$. Besides, these terms form the Aymptotic Bethe Ansatz, once the non-linear integrals are forgotten. No true systematics.
- Novelty:additional discontinuity equations on the cuts of the rapidity u-planes. We 'derived' the dressing factor from these relations (limitation of this 'explanation').

The Y-system

- It is the Y-system (not the TBA) which is encoded in a Dynkin-like diagram.
- I seat on a node: LHS= $Y_Q\left(u-\frac{i}{g}\right)Y_Q\left(u+\frac{i}{g}\right)=$
- RHS=Nearest neighbours products:
  - Horizontal: $\prod_{Q'}(1+Y_{Q'}(u))^{A_{QQ'}}$
  - Vertical: $\dfrac{\left(1+\frac{1}{Y^{(\alpha)}_{(v|Q-1)}(u)}\right)^{\delta_{Q,1}-1}}{\left(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}\right)^{\delta_{Q,1}}}$

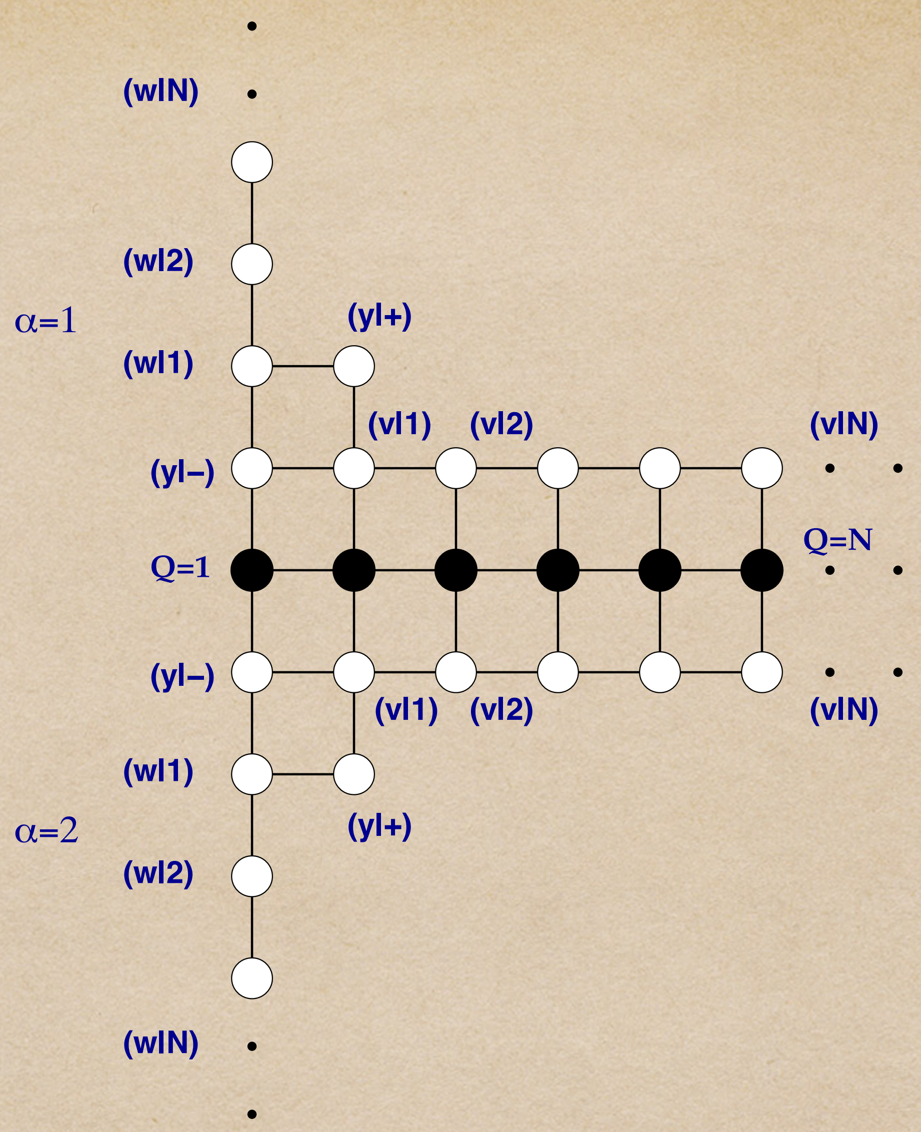


Figure 1: The Y-system diagram corresponding to the AdS_5/CFT_4 TBA equations.

$$Y_Q\left(u-\frac{i}{g}\right)Y_Q\left(u+\frac{i}{g}\right)=\prod_{Q'}\left(1+Y_{Q'}(u)\right)^{A_{QQ'}}\prod_{\alpha}\frac{\left(1+\frac{1}{Y^{(\alpha)}_{(v|Q-1)}(u)}\right)^{\delta_{Q,1}-1}}{\left(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}\right)^{\delta_{Q,1}}},$$

$$Y^{(\alpha)}_{(y|-)}\left(u+\frac{i}{g}\right)Y^{(\alpha)}_{(y|-)}\left(u-\frac{i}{g}\right)=\frac{\left(1+Y^{(\alpha)}_{(v|1)}(u)\right)}{\left(1+Y^{(\alpha)}_{(w|1)}(u)\right)}\frac{1}{\left(1+\frac{1}{Y_1(u)}\right)},$$

$$Y^{(\alpha)}_{(w|M)}\left(u+\frac{i}{g}\right)Y^{(\alpha)}_{(w|M)}\left(u-\frac{i}{g}\right)=\prod_N\left(1+Y^{(\alpha)}_{(w|N)}(u)\right)^{A_{MN}}\left[\frac{\left(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}\right)}{\left(1+\frac{1}{Y^{(\alpha)}_{(y|+)}(u)}\right)}\right]^{\delta_{M,1}},$$

$$Y^{(\alpha)}_{(v|M)}\left(u+\frac{i}{g}\right)Y^{(\alpha)}_{(v|M)}\left(u-\frac{i}{g}\right)=\frac{\prod_N\left(1+Y^{(\alpha)}_{(v|N)}(u)\right)^{A_{MN}}}{\left(1+\frac{1}{Y_{M+1}(u)}\right)}\left[\frac{\left(1+Y^{(\alpha)}_{(y|-)}(u)\right)}{\left(1+Y^{(\alpha)}_{(y|+)}(u)\right)}\right]^{\delta_{M,1}},$$

where $A_{1,M}=\delta_{2,M}$, $A_{NM}=\delta_{M,N+1}+\delta_{M,N-1}$ and $A_{MN}=A_{NM}$.

$$\Delta(u)=\left[\ln Y_1(u)\right]_{+1},\qquad [\Delta]_{\pm 2N}=\mp\sum_{\alpha=1,2}\left(\left[\ln\left(1+\frac{1}{Y^{(\alpha)}_{(y|\mp)}}\right)\right]_{\pm 2N}+\sum_{M=1}^{N}\left[\ln\left(1+\frac{1}{Y^{(\alpha)}_{(v|M)}}\right)\right]_{\pm(2N-M)}+\ln\left(\frac{Y^{(\alpha)}_{(y|-)}}{Y^{(\alpha)}_{(y|+)}}\right)\right),$$

$$\left[\ln\left(\frac{Y^{(\alpha)}_{(y|-)}}{Y^{(\alpha)}_{(y|+)}}\right)\right]_{\pm 2N}=-\sum_{Q=1}^{N}\left[\ln\left(1+\frac{1}{Y_Q}\right)\right]_{\pm(2N-Q)},$$

with $N=1,2,\ldots,\infty$ and

$$\left[\ln Y^{(\alpha)}_{(w|1)}\right]_{\pm 1}=\ln\left(\frac{1+1/Y^{(\alpha)}_{(y|-)}}{1+1/Y^{(\alpha)}_{(y|+)}}\right),\qquad \left[\ln Y^{(\alpha)}_{(v|1)}\right]_{\pm 1}=\ln\left(\frac{1+Y^{(\alpha)}_{(y|-)}}{1+Y^{(\alpha)}_{(y|+)}}\right),$$

where the symbol $[f]_Z$ with $Z\in\mathbb{Z}$ denotes the discontinuity of $f(z)$

$$[f]_Z=\lim_{\epsilon\to 0^+} f(u+iZ/g+i\epsilon)-f(u+iZ/g-i\epsilon),$$

on the semi-infinite segments described by $z=u+iZ/g$ with $u\in(-\infty,-2)\cup(2,+\infty)$
function $[f(u)]_Z$ is the analytic extension of the discontinuity

$$\Rightarrow\quad \left[f(u)\right]_Z=f(u+iZ/g)-f(u_*+iZ/g)$$

ODE/IM Correspondence: a quick review (Dorey,Tateo,BLZ,Dunning,Suzuki,Frenkel,Bender.....)

- Simplest example: Schroedinger eq. on the half line $(0,\infty)$ (Stokes line)
- $$\left(-\frac{d^2}{dx^2}+x^{2M}+\frac{l(l+1)}{x^2}\right)\psi(x)=E\psi(x)$$
- we fix the subdominant solution such that at complex infinity
- $$y\sim x^{-M/2}\exp\left(-\frac{1}{M+1}x^{M+1}\right),$$ $$y'\sim -x^{M/2}\exp\left(-\frac{1}{M+1}x^{M+1}\right)$$

$$|\arg x|<\frac{3\pi}{2M+2}$$

- Changing anti-Stokes sector $\mathscr{S}_k = \left|\arg x-\frac{2k\pi}{2M+2}\right|<\frac{\pi}{2M+2}$ this solution becomes dominant

Discrete Symmetry Breaking

- Omega symmetry of the eq. not of the solution which rotates by $\omega = \exp(\pi i/(M+1)) = q$, quantum group
- $\hat{\Omega}: x \to qx, \quad E \to q^{-2}E, \quad l \to l \qquad y_k \equiv y_k(x,E,l) = \omega^{k/2} y(\omega^{-k}x, \omega^{2k}E, l)$
- y_k subdominant in $\mathcal{S}_k$ and dominant in $\mathcal{S}_{k \pm 1}$
- around infinity, irregular singularity.
- Lambda symmetry, around zero, regular singularity:
- $\hat{\Lambda}: x \to x, \quad E \to E, \quad l \to -1-l \qquad \hat{\Lambda}\psi^{\pm} = \psi^{\mp} \qquad \psi^{+}(x,E,l) \sim x^{l+1} + O(x^{l+3})$

Transfer matrix T, Q and various functional equations

- Stokes multipliers
- $$y_{k-1}(x,E,l) = C_k(E,l)\, y_k(x,E,l) + \tilde{C}_k(E,l)\, y_{k+1}(x,E,l)$$
- Wronskian interpretation, k=0
- $$C = \frac{W_{-1,1}}{W_{0,1}}, \qquad \tilde{C} = -\frac{W_{-1,0}}{W_{0,1}}$$
- essentially by using the leading asymptotics
- all of the $\tilde{C}_k$ are identically equal to -1 $\quad C(E,l) = \frac{1}{2i} W_{-1,1}(E,l)$
- $$C(E,l)\, y(x,E,l) = \omega^{-1/2} y(\omega x, \omega^{-2} E, l) + \omega^{1/2} y(\omega^{-1} x, \omega^2 E, l)$$

- If l=0, no singularity in x=0, then Baxter TQ-relation
- $$\mathbf{T}(\lambda)\mathbf{Q}_{\pm}(\lambda) = \mathbf{Q}_{\pm}(q^{-1}\lambda) + \mathbf{Q}_{\pm}(q\lambda)$$
- but keeping $l \neq 0$, I would expect
- $$C(E,l)D^{\mp}(E,l) = \omega^{\mp(1/2+l)}D^{\mp}(\omega^{-2}E,l) + \omega^{\pm(1/2+l)}D^{\mp}(\omega^{2}E,l)$$
- In fact transport (Jost) coefficients
- $$D^{\mp}(E,l) \equiv W\left[\, y(x,E,l), \psi^{\pm}(x,E,l)\right]$$
- are projections on the psi. Scattering theory $(0,\infty)$

- From the TQ relation or the QQ-system (more fundamental), n=0 (n=1 definition of T) of
- $$(4l+2)\,i\,C^{(n)}(E) = \omega^{(n+1)(l+1/2)}D^{-}(\omega^{n+1}E,l)\,D^{+}(\omega^{-n-1}E,l) - \omega^{-(n+1)(l+1/2)}D^{-}(\omega^{-n-1}E,l)\,D^{+}(\omega^{n+1}E,l)$$
- the whole integrability machinery develops functional equations; here we just need pay attention to their derivation/interpretation from the ODE
- Fused T relations
- $$\mathbf{T}(\lambda)\mathbf{T}_j(q^{j+1/2}\lambda) = \mathbf{T}_{j-1/2}(q^{j+1}\lambda) + \mathbf{T}_{j+1/2}(q^{j}\lambda)$$

  or

  $$\mathbf{T}(\lambda)\mathbf{T}_j(q^{-j-1/2}\lambda) = \mathbf{T}_{j-1/2}(q^{-j-1}\lambda) + \mathbf{T}_{j+1/2}(q^{-j}\lambda)$$
- which brings the TT-system or discrete Hirota eq.
- $$\mathbf{T}_j(q^{-1/2}\lambda)\mathbf{T}_j(q^{1/2}\lambda) = \mathbf{1} + \mathbf{T}_{j+1/2}(\lambda)\mathbf{T}_{j-1/2}(\lambda)$$
- with the ODE identification with the Wronskian $T_{n/2}(\nu E^{1/2}) = C^{(n)}(E) = \frac{1}{2i}W_{-1,n}(\omega^{-n+1}E)$.

- Finally the Y-system for the gauge invariant quantity
- $$Y_n(E) = C^{n+1}(E)C^{n-1}(E)$$
- which easily brings the T-system into the form
- $$Y_n(\omega E)Y_n(\omega^{-1}E) = (1+Y_{n+1}(E))(1+Y_{n-1}(E))$$
- Upon inverting the shift operator on the l.h.s., and using a suitable asymptotic as zero-mode, we can obtain non-linear integral equations with universal kernel 1/cosh, equivalent to physical TBA eqs.

2D CFT dictionary

- Eigenvalues of statistical mechanics operators Q and T on the conformal primary (dimension)

- $$\Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}. \qquad p = \frac{2l+1}{4M+4}$$

- with 'minimal model' central charge

$$c = 13 - 6\left(\beta^2 + \beta^{-2}\right) \qquad \beta^2 = \frac{1}{M+1}$$

Sine-Gordon coupling

$$q = e^{i\pi\beta^2}$$

T, Q and the SW-NS periods (DF, D. Gregori)

- Via AGT correspondence we quantise/deform the quadratic SW differential by the level 2 null vector eq. (Mathieu)
- $$-\frac{\hbar^2}{2}\frac{d^2}{dz^2}\psi(z)+[\Lambda^2\cos z-u]\psi(z)=0$$
- Namely, quantum SW differential $\mathcal{P}(z)=-i\frac{d}{dz}\ln\psi(z)$ and periods
- $$a(\hbar,u,\Lambda)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}(z;\hbar,u,\Lambda)\,dz\ ,\qquad a_D(\hbar,u,\Lambda)=\frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos(u/\Lambda^2)-i0}\mathcal{P}(z;\hbar,u,\Lambda)\,dz$$
- ODE/IM treatment of this eq. goes its non-compact (modified) version: two irregular singularities (M=-2)
- $$\left\{-\frac{d^2}{dy^2}+2e^{2\theta}\cosh y+P^2\right\}\psi(y)=0$$
- Gauge/integrability change of variable
- $$\frac{\hbar}{\Lambda}=e^{-\theta}\ ,\qquad \frac{u}{\Lambda^2}=\frac{P^2}{2e^{2\theta}}$$

- Integrability/gauge identification
- $$T(\hbar, u, \Lambda) \equiv T(\theta, P^2) = 2\cos\{2\pi a(\hbar, u, \Lambda)\}$$
- $$Q(\theta, P^2) \equiv Q(\hbar, u, \Lambda) = \exp\left\{2\pi i a_D(\hbar, u, \Lambda)\right\}$$
- The $Y = Q^2$ system
- $$1+Q^2(\theta,P^2) = Q(\theta-i\pi/2,P^2)Q(\theta+i\pi/2,P^2)\,, \qquad 1+Q^2(\theta,u) = Q(\theta-i\pi/2,-u)Q(\theta+i\pi/2,-u)$$
- from which TBA eq.
- $$\varepsilon(\theta,u,\Lambda) = -4\pi i a_D^{(0)}(u,\Lambda)\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln[1+\exp\{-\varepsilon(\theta',-u,\Lambda)\}]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}$$
$$\varepsilon(\theta,-u,\Lambda) = -4\pi i \underbrace{a_D^{(0)}(-u,\Lambda)}_{\text{dyon}}\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln[1+\exp\{-\varepsilon(\theta',u,\Lambda)\}]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}$$
- TQ-system and periodicity of T are the quantum Bilal-Ferrari ($u \to -u$ symmetry breaking)
- $$T(\theta,P^2) = \frac{Q(\theta-i\pi/2,P^2)+Q(\theta+i\pi/2,P^2)}{Q(\theta,P^2)}\,, \qquad T(\theta,u) = \frac{Q(\theta-i\pi/2,-u)+Q(\theta+i\pi/2,-u)}{Q(\theta,u)}$$
- asymptotic expansion into quantum periods (n=0 is SW)
- $$a_D^{(n)}(-u) = i(-1)^n\left[-\operatorname{sgn}(\operatorname{Im} u)\, a_D^{(n)}(u) + a^{(n)}(u)\right]$$

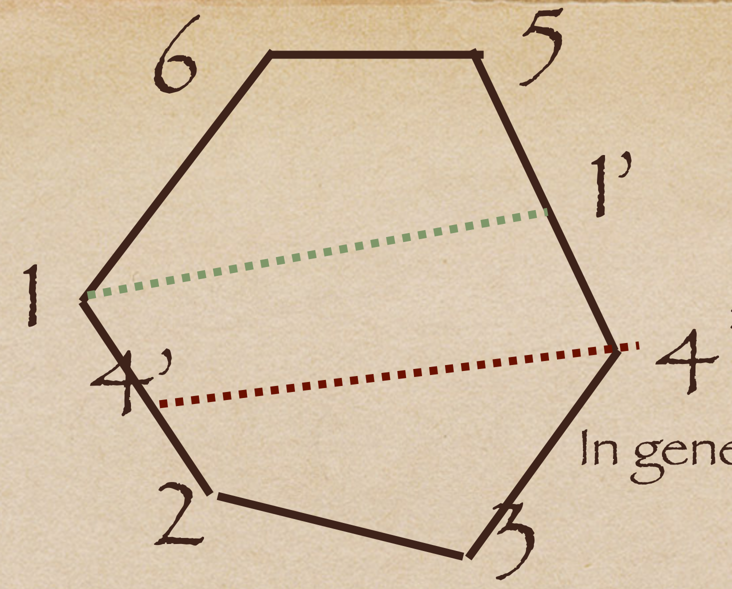
- Unexpected surprise
- $$\left\{-\frac{d^2}{dy^2} + e^{2\theta}(e^{y/b} + e^{-yb}) + P^2\right\}\psi(y) = 0$$
- previous eq. is the $b = 1$ case describes Liouville field theory vacua
- $\Delta = (c-1)/24 - P^2 \qquad c = 1 + 6(b + b^{-1})^2$
- Self-dual point of the symmetry $b \to 1/b$! And somehow previous $\beta = ib$
- Coincidence? Meaning of this Liouville field theory?

A third way to TBA: the OPE for null polygonal WLs

- Theory: N=4 SYM in planar limit $\lambda = N_c g^2_{YM},\, N_c \to \infty$
- Dual to quantum area of II B string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$
- Light-like polygons can be decomposed into light-like Pentagons (and Squares): an Operator Product Expansion
- Prototype: Hexagon into two Pentagons P
- The same as two-point correlation function <PP> into Form-Factors in quantum integrable 2D field theories

- In a picture:

hexagon



=P(12341') P(14'456)

In general: E-5 shared squares, E-4 pentagons

- Which mathematically means:

    - W=Σ exp(-rE)<0|P|n><n|P|0>

Multi-P correlation function:general m,n transition

- =<PP>: the same as 2D Form Factor (FF) decomposition

- Form-Factors obey axioms with the S-matrix: 1)Watson eqs., 2) Monodromy (q-KZ), 3) Kinematic Poles, 4) Bound-state eqs. etc.

- We had to modify the 2) (and 3)) (for twist fields)

- Eigen-states |n>? 2D excitations over the GKP folded string (of length=2 ln s) which stretches from the boundary to boundary (for large s) of AdS.

- The quantum GKP string can be represented by the quantum spin chain vacuum (gauge)

$$\Omega_{GKP} = Tr\, Z D_+^s Z + \dots$$

- 2D particles: 6 scalars, 2 gluons, 4+4 (anti)fermions Bethe states:

$$\mathcal{O}_{1-particle} = Tr\, Z D_+^{s-s'} \varphi D_+^{s'} Z + \dots$$

$\varphi = Z, W, X, F_{+\perp}, \bar{F}_{+\perp}, \Psi_+, \bar{\Psi}_+$ Dispersion relation

- Scattering over the GKP vacuum:

$$\mathcal{O}_{2-particles} = Tr\, Z D_+^{s-s_1-s_2} \varphi_1 D_+^{s_1} \varphi_2 D_+^{s_1} Z + \dots$$

- Two-body is enough because of integrability

FFs series summing to TBA

- Quite unique example of Form-Factor series re-summation. Result: thermodynamic bubble Ansatz of string minimal area at strong coupling (Alday-Gaiotto-Maldacena)

- The key idea: Hubbard-Stratonovich transformation replaces the infinite sums with a path integral

$$W_{hex}^{(g)} = Z^{(g)}[X^g] = \int \mathcal{D}X^g e^{-S^{(g)}[X^g]}$$

$$S^{(g)}[X^g] = \frac{1}{2}\int d\theta\, d\theta'\, X^g(\theta) T^g(\theta,\theta') X^g(\theta') +$$
$$+ \int \frac{d\theta'}{2\pi}\, \mu^g(\theta') \left[\mathrm{Li}_2(-e^{-E(\theta')+i\phi}\, e^{X^g(\theta')}) + \mathrm{Li}_2(-e^{-E(\theta')-i\phi}\, e^{X^g(\theta')})\right]$$

- $S^{(g)}[X^g] \sim \sqrt{\lambda} \to \infty$: saddle point eqs. are TBA eqs.

$$X^g(\theta) - \int \frac{d\theta'}{2\pi}\, G^g(\theta,\theta')\mu^g(\theta')\, \log\left[(1+e^{X^g(\theta')}e^{-E(\theta')+i\phi})(1+e^{X^g(\theta')}e^{-E(\theta')-i\phi})\right] = 0$$

$$\int d\theta'\, G^g(\theta,\theta') T^g(\theta',\theta'') = \delta(\theta-\theta'')$$

- For the simplest hexagon, equivalent to the A3 TBA (Al. Zamolodchikov).
- We also reproduced the general E-gon: A3x(E-5 columns): delicate determination of the convolution integration contours
- We reproduced TBA with only gluons and 'mesons' (world-sheet meson is a 2D fermion-antifermion bound state only at strong coupling, other particle contribution is superficially 1-loop)
- New way to consider: 1) TBA from spectral series which gives rise to a Yang-Yang functional (=area) (similar to how it arises in N=2 SYM (Nekrasov-Shatashvili)); but here 2) PDE/quantum Integrable Model, PDE is a classical Lax pair.
- Very recently we have found ODE/IM also for NS regime.
- Weak coupling (gauge) results: tree level and 1-loop (Basso,Sever,Vieira+Perimeter). 2-loops (Dixon,Drummond et al.) by using field theory methods.

Scalars contribution scales as

$$\ln W = \frac{\sqrt{\lambda}}{\pi} \sum_{n=1}^{+\infty} \frac{1}{(2n)!} \int \prod_{i=1}^{2n-1} \frac{d\alpha_i}{2\pi} g^{(2n)}(\alpha_1, \ldots, \alpha_{2n-1}) + O(\ln \sqrt{\lambda})$$

- the same order as the classical minimal area: $-\frac{\sqrt{\lambda}}{2\pi} A_E$
- Check with Knizhnik twist field dimension

$$\Delta_\alpha = \frac{c}{12}(k - 1/k), \;\; \alpha = 2\pi k - 2\pi = \pi/2, c = 5$$

- and we can also compute beyond leading: new feature is divergency (asymptotic freedom of O(6) NL Sigma Model).

Some Perspectives

- Non-linear integral or functional equations are powerful and are the monodromies of a ODE or PDE. There is any deep reason why these (TBA) are reproduced by an integrable Form Factor series of a 'weird' scattering theory?
- Saddle point: classical string $\longrightarrow$ Quantisation? Quantum PDE/IM? q-TBA?
- NS limit $\epsilon_2 = 0$: ODE/IM $\epsilon_1 = \hbar$ $\longrightarrow$ $\epsilon_2 \neq 0$: quantum ODE/IM?
- On the contrary: meaning of $b \neq 1$ of our Liouville field theory (not AGT)?
- Formal similarity between OPE series and N=2 (Nekrasov) partition function: e.g. ADHM set-up: meaning? With Poghossians.

Thanks